INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 1.8]

COMPLEX NUMBERS (CARTESIAN FORM)

Compiled by Christos Nikolaidis

A. Paper 1 questions (SHORT)

1. [Maximum mark: 7]

Let $f(z) = z^2 - 8z + 20$.

(a) Find the discriminant Δ of the quadratic function f. [2]

(b) Find the complex roots of the equation $f(z) = 0$ in the form $z = a \pm b\mathrm{i}$ by using the formula $z = \dfrac{-b \pm \mathrm{i}\sqrt{|\Delta|}}{2a}$ [3]

(c) Use factorisation to express f in the form $f(z) = (z-h)^2 + k$. [2]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

2. [Maximum mark: 8]

The complex roots of the equation $az^2 + bz + c = 0$ are given by $z = \dfrac{-b \pm \mathrm{i}\sqrt{|\Delta|}}{2a}$.

where $\Delta = b^2 - 4ac$

(a) Find the complex roots of the equation $4z^2 - 8z + 13 = 0$ by using the given formulas by expressing your answers in the form $z = x \pm y\mathrm{i}$. [4]

(b) Confirm that the sum S and the product P of the roots are given by

(i) $S = -\dfrac{b}{a}$. (ii) $P = \dfrac{c}{a}$. [4]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

3. [Maximum mark: 6]

Find **analytically**

(a) $(1-\mathrm{i}\sqrt{3})^2$ in the form $a+b\mathrm{i}$, where $a,b \in Z$. [3]

(b) $(1-\mathrm{i}\sqrt{3})^3$. [3]

[Confirm the results by your GDC]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

4. [Maximum mark: 6]

Let the complex number z be given by $z = 1 + \dfrac{\mathrm{i}}{\mathrm{i}-\sqrt{3}}$.

Express z **analytically** in the form $a+b\mathrm{i}$, giving the **exact** values of a,b.

[Confirm the result by your GDC]

.....

.....

.....

.....

.....

.....

.....

.....

.....

5. [Maximum mark: 5]

Let $z = \frac{2}{1-\mathrm{i}} + 1 - 4\mathrm{i}$, Express z^2 in the form $x + y\mathrm{i}$ where $x, y \in Z$.

Show **analytically** your work.

[Confirm the result by your GDC]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

6. [Maximum mark: 4]

(a) Given that $(a-2)+3\mathrm{i} = 7+(b-1)\mathrm{i}$, find the value of a and of b, where $a, b \in \mathbb{Z}$. [2]

(b) Given that $(c-2)+(d-1)\mathrm{i} = 0$, find the value of c and of d, where $c, d \in \mathbb{Z}$. [2]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

7. [Maximum mark: 7]

Let $z = x + y\mathrm{i}$. Find the values of x and y if

$$(2+5\mathrm{i})z = 1+17\mathrm{i}$$

(a) by substituting $z = x + y\mathrm{i}$ in the equation and solving the simultaneous equations. [3]

(b) by using division (an equation of the form $az = b$, implies $z = \frac{b}{a}$). [4]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

8. [Maximum mark: 6]

Given that $(a+\mathrm{i})(2-b\mathrm{i})=7-\mathrm{i}$, find the value of a and of b, where $a,b\in\mathbb{Z}$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

9. [Maximum mark: 6]

Consider the equation $2(p+\mathrm{i}q)=q-\mathrm{i}p-2(1-\mathrm{i})$, where p and q are both real numbers. Find p and q.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

10. [Maximum mark: 4]

Let $z = x + y\mathrm{i}$. Find the values of x and y if $(1-\mathrm{i})z = 1-3\mathrm{i}$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

11. [Maximum mark: 4]

Find the values of a and b, where a and b are real, given that $(a+b\mathrm{i})(2-\mathrm{i}) = 5-\mathrm{i}$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

12. [Maximum mark: 6]

Solve the following equation for z, where z is a complex number.

$$\frac{z}{3+4\mathrm{i}}+\frac{z-1}{5\mathrm{i}}=\frac{5}{3-4\mathrm{i}}$$

Give your answer in the form $a+b\mathrm{i}$ where $a,b\in Z$.

13. [Maximum mark: 6]

The two complex numbers $z_1 = \frac{a}{1+\mathrm{i}}$ and $z_2 = \frac{b}{1-2\mathrm{i}}$ where $a, b \in R$, are such that $z_1 + z_2 = 3$. Calculate the value of a and of b.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

14. [Maximum mark: 5]

The complex number z satisfies $\mathrm{i}(z+2) = 1 - 2z$. Write z in the form $z = a + b\mathrm{i}$, where a and b are real numbers.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

15. [Maximum mark: 6]

Let z_1 and z_2 be complex numbers. Solve **analytically** the simultaneous equations

$$2z_1 + 3z_2 = 7$$
$$z_1 + \mathrm{i}z_2 = 4 + 4\mathrm{i}$$

Give your answers in the form $z = a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$.

[Confirm the result by your GDC]

16. [Maximum mark: 6]

Given that $|z| = 2\sqrt{5}$, find the complex number z that satisfies the equation

$$\frac{25}{z} - \frac{15}{z^*} = 1 - 8\mathrm{i}.$$

17. [Maximum mark: 7]

Let $w = \dfrac{z}{z^2+1}$ where $z = x + \mathrm{i}y$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that w is a real number show that $|z| = 1$.

18. [Maximum mark: 8]

(a) Evaluate $(1+\mathrm{i})^2$, where $\mathrm{i}^2=-1$. [2]

(b) Find $(a+a\mathrm{i})^2$ in terms of a and i. [2]

(c) Find $(a+a\mathrm{i})^4$ in terms of a. [2]

(d) Find $(a+a\mathrm{i})^8$ in terms of a. [2]

B. Paper 2 questions (LONG)

19. [Maximum mark: 14]

Find the complex number z in each of the following equations.

(a) $(2+5\mathrm{i})z+9=3+14\mathrm{i}$ [3]

(b) $(2+5\mathrm{i})z+9=3z+19\mathrm{i}$ [5]

(c) $(2+5\mathrm{i})z+8=3\bar{z}+20\mathrm{i}$ [6]

20. [Maximum mark: 12]

Given that $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$, show that

(a) Express zw in the form $x + y\mathrm{i}$ [2]

(b) Show that $|zw|^2 = (ac)^2 + (bd)^2 + (ac)^2 + (bd)^2$ [2]

(c) Show that

(i) $\overline{z + w} = \bar{z} + \bar{w}$ (ii) $\overline{zw} = \bar{z}\,\bar{w}$ (iii) $|zw| = |z||w|$ [8]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

21. [Maximum mark: 12]

Let $z = x + y\mathrm{i}$ be a complex number.

(a) Given that $w = z + 16$

(i) write down the conjugate of w

(ii) express $|w|^2$ in the form $x^2 + y^2 + ax + b$. [5]

(b) Express $|z+1|^2$ in terms of x and y. [2]

(c) Given that $|z+16| = 4|z+1|$, find the value of $|z|$. [5]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....